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The Slow Passage through a Steady Bifurcation: Delay and Memory Effects

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We consider the following problem as a model for the slow passage through a steady bifurcation: $dy/dt = \lambda(t)\, y - y^3 + \delta$, where λ is a slowly increasing function of t given by $\lambda = \lambda_i + \varepsilon t$ ($\lambda_i < 0$). Both ε and δ are small parameters. This problem is motivated by laser experiments as well as theoretical studies of laser problems. In addition, this equation is a typical amplitude equation for imperfect steady bifurcations with cubic nonlinearities. When $\delta = 0$, we have found that $\lambda = 0$ is not the point where the bifurcation transition is observed. This transition appears at a value $\lambda = \lambda_j > 0$. We call λ_j the delay of the bifurcation transition. We study this delay as a function of λ_i, the initial position of λ, and δ, the imperfection parameter. To this end, we propose an asymptotic study of this equation as $\delta \to 0$, ε small but fixed. Our main objective is to describe this delay in terms of the relative magnitude of δ and ε. Since time-dependent imperfections are always present in experiments, we analyze in the second part of the paper the effect of a small-amplitude but time-periodic imperfection given by $\delta(t) = \delta \cos(\sigma t)$.

KEY WORDS: Slowly varying bifurcation parameter; effects of steady and time-periodic imperfections; control of laser instabilities.

1. INTRODUCTION

In many experiments that are modeled as bifurcation problems the bifurcation parameter is deliberately varied in time to sweep across a whole portion of the bifurcation diagram.[1-9] This procedure is usually justified by the assumption that if the sweep rate is small enough, the bias introduced by a time-dependent parameter should not modify qualitatively

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the static bifurcation diagram corresponding to a truly constant bifurcation parameter.

Previous studies of such bifurcation problems[10-17] indicate that the transfer of stability expected at a bifurcation or at a limit point may be delayed by the time dependence of the bifurcation parameter. This delay is controlled, in particular, by (1) the nature of the static critical point (steady bifurcation, Hopf bifurcation or limit point); (2) the initial value of the bifurcation parameter (since the problem is now nonautonomous); (3) the presence of small imperfections (static imperfections or noise). Thus, the delay produced by a slowly varying control parameter is a complicated function of several key parameters. To determine how the delay depends on the external parameters, we shall analyze a simple bifurcation problem using asymptotic methods. Such a detailed study can be valuable in connection with careful experiments designed to find out how to control (i.e., decrease or increase) this delay effect. This kind of information can also be obtained numerically, but the use of asymptotic results may be easier because the nonautonomous bifurcation problem depends on several time scales.

From the physical point of view, a slowly varying parameter is an alternative mechanism for the experimental determination of the bifurcation diagram of the stable solutions. This implies, however, a good understanding of the perturbation produced by slow variations of the bifurcation parameter. In particular, we must analyze the slow passage through the critical point where transients are slow and the deviation between the exact solution and the static bifurcation diagram is expected to be maximum.

In this paper, we pursue our study of the following imperfect bifurcation problem:

$$y_t = \lambda(t)\, y - y^3 + \delta \tag{1.1}$$

$$\lambda(t) = \lambda_i + \varepsilon t \tag{1.2}$$

$$y(0) = y_i > 0, \qquad \lambda_i < 0, \quad \varepsilon, \delta > 0 \tag{1.3}$$

The nonlinear first-order equation (1.1) is a typical model problem for laser instabilities.[15] Assuming $\lambda_i = O(1)$ and $y_i = O(1)$, we find that the problem depends on two parameters: $\lambda(t)$ is the slowly varying control or bifurcation parameter and δ is the imperfection parameter. Both ε and δ are small quantities. The behavior of the solutions of (1.1)–(1.3) depends on the relative values of δ and ε.[15] Of particular physical interest is the case $\delta \ll \varepsilon$ because the transition near the bifurcation transition is considerably delayed. A typical example is shown in Fig. 1 for $\delta = 10^{-3}$ and $\varepsilon = 0.1$: y is almost zero and is slowly varying until a jump transition to another slowly

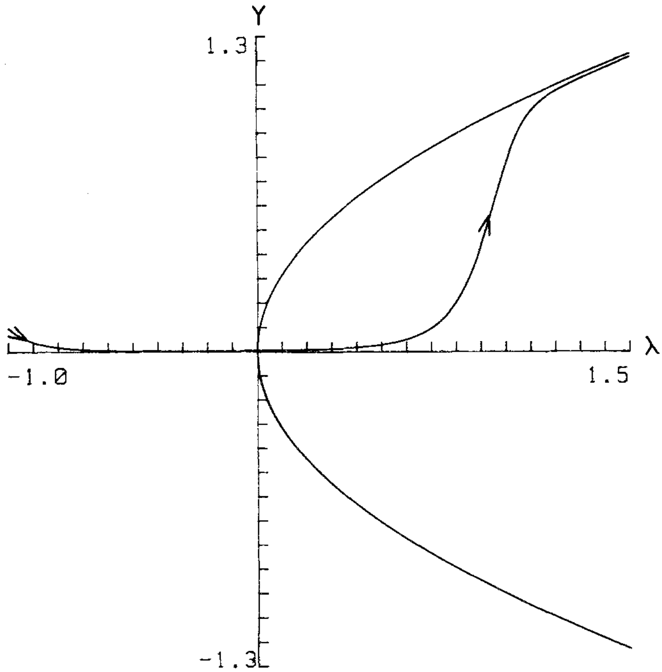

Fig. 1. Example of delayed bifurcation in the presence of an imperfection. Comparison between the perfect static curve $y^2 = \lambda$ and the solution of $y_t = \lambda(t)\, y - y^3 + \delta$ (curve with the arrows). Parameters are $\delta = 10^{-3}$, $\lambda = -1 + 10^{-1}t$, and $y(0) = 0.1$.

varying solution occurs at $\lambda = \lambda_j > 0$. The value of λ_j is the delay of the bifurcation transition. Our main purpose is to determine $\lambda_j(\varepsilon, \delta)$ analytically. To this end, we shall investigate the limit $\delta \to 0$, ε fixed.

Recently, the effect of small-amplitude noise on the time-dependent bifurcation diagram was investigated for a laser problem and an electronic system.[18-20] The results of this study raise the question of the persistence of the delayed transition when the imperfection itself is time-dependent. As a first approach to this problem, we shall analyze a model equation where the imperfection is periodic in time. Specifically, we consider the modified equation

$$y_t = \lambda(t)\, y - y^3 + \delta \cos(\sigma t) \tag{1.4}$$

which now depends on three parameters: λ, δ, and σ, the frequency of the time-periodic imperfection. Clearly, as $\sigma \to 0$, $\cos(\sigma t) \sim 1$ if $t \ll 1/\sigma$ and problem (1.4) reduces to Eq. (1.1). On the other hand, if $\sigma \to \infty$, the time-periodic perturbation operates on a fast time scale and only its averaged value will contribute to the long-time response of Eq. (1.4). This average value is zero. Consequently, we expect that the solution of Eq. (1.4) approaches the solution of problem (1.1) without imperfections ($\delta = 0$). Thus, σ is an important parameter for the delayed bifurcation transition and will considerably affect the position λ_j of the jump transition to the new state.

2. CONSTANT IMPERFECTION

We introduce a new time variable τ defined by $\tau = \lambda(t)$. This specifies $\tau = 0$ to be the instant at which the static bifurcation point $\lambda = 0$ is reached. In terms of τ, Eqs. (1.1)–(1.3) can be rewritten as

$$\varepsilon y_\tau = \tau y - y^3 + \delta \tag{2.1}$$

$$y(\lambda_i) = y_i > 0, \qquad \lambda_i < 0 \tag{2.2}$$

This equation was analyzed in Ref. 15. When δ is fixed and $\varepsilon \to 0$, we found the following results:

1. There exists a stable, slowly varying solution on the τ/ε scale given by

$$y_{slv}(\tau, \varepsilon) = y_s(\tau) + O(\varepsilon) \tag{2.3}$$

where $y_s(\tau)$ is implicitly given by

$$\tau = y_s^2 - \delta/y_s, \qquad y_s > 0 \tag{2.4}$$

if δ is sufficiently large (i.e., if $\delta \gg \varepsilon^{3/4}$), the solution of Eq. (2.1) and (2.2) quickly approaches the slowly varying solution (2.3) and the transition near $\tau = 0$ is smooth and continuous. This regime corresponds to an "adiabatic following" of the static steady state:

$$\lambda = y_s^2 - \delta/y_s, \qquad y_s > 0, \quad \lambda_t = 0$$

2. If $\delta = O(\varepsilon^{3/4})$, the expansion (2.3) becomes nonuniform near $\tau = 0$. The solution of Eq. (2.1) still approaches a slowly varying regime, which slightly deviates from (2.3) when $|\tau| = O(\varepsilon^{1/2})$. As δ is further decreased, this deviation increases and becomes maximum as $\delta \to 0$.

We now analyze the limit $\delta \to 0$ in detail. To this end, we seek an asymptotic expansion as $\delta \to 0$ of the solution of (2.1) and (2.2) in the form

$$y(\tau, \delta, \varepsilon) = y_0(\tau, \varepsilon) + \delta y_1(\tau, \varepsilon) + \delta^2 y_2(\tau, \varepsilon) + \cdots \tag{2.5}$$

The coefficients y_0, y_1, y_2 are determined by inserting (2.5) into (2.1) and equating to zero the resulting coefficients of each power of δ. We then obtain a hierarchy of problems for $y_0, y_1, \ldots$. The equations for y_0 and y_1 are given by

$$\varepsilon y_{0\tau} = \tau y_0 - y_0^3, \qquad y_0(\lambda_i) = y_i \tag{2.6}$$

$$\varepsilon y_{1\tau} = (\tau - 3y_0^2)\, y_1 + 1, \qquad y_1(\lambda_i) = 0 \tag{2.7}$$

Equation (2.6) is a Bernoulli equation and can be solved exactly. As $\varepsilon \to 0$ and provided that $\lambda_i < \tau < -\lambda_i$, y_0 is an exponentially small function given by

$$y_0 \sim A \exp[(\tau^2 - \lambda_i^2)/2\varepsilon] \ll 1 \qquad \text{as} \quad \varepsilon \to 0 \tag{2.8a}$$

where

$$A = y_i(1 - y_i^2/\lambda_i)^{-1/2} \tag{2.8b}$$

Using (2.8), we then solve Eq. (2.7) for y_1. Considering now the time interval $0 < \tau < -\lambda_i$, we obtain

$$y_1 \sim (2\pi/\varepsilon)^{1/2} \exp(\tau^2/2\varepsilon) \qquad \text{as} \quad \varepsilon \to 0 \tag{2.9}$$

which increases exponentially as soon as τ increases. Thus, in order to keep the expansion (2.5) uniform for the interval $0 < \tau < -\lambda_i$, δ must be appropriately small. Let

$$\delta = \delta_0(\varepsilon) \exp(-k^2/2\varepsilon) \tag{2.10a}$$

where k is assumed to be an $O(1)$ constant that characterizes the smallness of δ and $\delta_0(\varepsilon)$ is a fixed algebraic function of ε,

$$\delta_0(\varepsilon) = \varepsilon^p \tag{2.10b}$$

where $p > 0$. Using (2.8)–(2.10), we find that the expansion (2.5) now reads

$$y(\tau, \delta) \sim A \exp[(\tau^2 - \lambda_i^2)/2\varepsilon] + \delta_0(\varepsilon)(2\pi/\varepsilon)^{1/2} \exp[(\tau^2 - k^2)/2\varepsilon] + O(\delta^2) \qquad \text{as} \quad \varepsilon \to 0 \tag{2.11}$$

and for the interval $0 < \tau < -\lambda_i$. Considering the first two terms of this expansion, we note that it remains uniform provided that

$$\exp[(\lambda_i^2 - k^2)/2\varepsilon] < (\varepsilon/2\pi)^{1/2} A/\delta_0(\varepsilon) \tag{2.12}$$

If $k > -\lambda_i$, this condition is always fulfilled. If $k < -\lambda_i$, (2.12) requires that $\lambda_i^2 - k^2 < -(2p-1)\varepsilon \ln \varepsilon$ and $p > 1/2$. The analysis of the higher order correction of the solutions, i.e., $\delta^2 y_2(\tau)$, leads to the same conclusion for the validity of the expansion (2.5). The expression (2.11) represents a nearly zero solution until τ approaches $-\lambda_i$ or k. Beyond that point, the solution diverges, which corresponds to the beginning of a transition toward the vicinity of another steady solution. There exist two possibilities, depending on the reltive values of k and $-\lambda_i$.

(i) If $k > -\lambda_i$, the first term in (2.11) increases exponentially as $\tau \sim -\lambda_i$.

(ii) If $k < -\lambda_i$, the second term in (2.11) dominate as $\tau \sim k$.

A dynamical behavior described by two competing exponentials is not new in singular perturbation problems. It has been introduced for other problems[21,22] and seems to be associated with the perturbation of separatrix trajectories (here the solution $y = 0$ when $\delta = 0$).

In summary, the results of this section suggest that there exist two distinct behaviors for the delayed bifurcation transitions. They mainly depend on the relative values of δ and ε:

1. If $\delta < \delta_c$, where $\delta_c = \varepsilon^p \exp(-\lambda_i^2/2\varepsilon)$, the delay is maximum and is given by $\tau \sim -\lambda_i$. It is independent of the value of δ and is a function of λ_i.
2. If $\delta > \delta_c$, the jump transition appears near $\tau \sim k$ and is a function of δ, but not of λ_i.

As $\delta \to O(\varepsilon^{3/4})$, the delay becomes insignificant. The qualitative predictions of this analysis are examined numerically in Section 4.

3. TIME-PERIODIC IMPERFECTION

In this section, we concentrate on Eqs. (1.2)–(1.4) and analyze the effect of a time-periodic imperfection. Sefining $\tau = \lambda(t)$ as our basic time variable, we can rewrite these equations as

$$\varepsilon y_\tau = \tau y - y^3 + \delta \cos[\sigma(\tau - \lambda_i)/\varepsilon] \tag{3.1}$$

$$y(\lambda_i) = y_i > 0, \qquad \lambda_i < 0 \tag{3.2}$$

As discussed in the introduction, we expect different responses, depending on the relative values of σ, δ, and ε. We consider several cases.

3.1. $\sigma < \epsilon$

If σ is sufficiently small compared to ε, the imperfection is quasiconstant during the interval $\lambda_i < \tau < -\lambda_i$. Thus, the delayed transition will depend on the relative values of δ and ε as analyzed in Section 2.

3.2. $\sigma = O(\epsilon)$

If $\sigma = O(\varepsilon)$, the periodic forcing operates on the same timescale as the slowly varying control parameter and perturbs the slow passage through the bifurcation point. We first define

$$\omega = \sigma/\varepsilon \tag{3.3}$$

as a specified $O(1)$ parameter. Assuming $y_i = O(1)$, we then seek a solution of (3.1) and (3.2) of the form (2.5). After substituting (2.5) and (3.3) into (3.1) and (3.2), we find that y_0 satisfies Eq. (2.6) and that the equation for y_1 is given by

$$\begin{aligned} \varepsilon y_{1\tau} &= (\tau - 3y_0^2)\, y_1 + \cos[\omega(\tau - \lambda_i)] \\ y_1(\lambda_i) &= 0 \end{aligned} \tag{3.4}$$

If $\tau < -\lambda_i$, y_0 is given by (2.8) and is exponentially small. Then solving (3.4) with $y_0 \ll 1$, we obtain

$$y_1(\tau) = \varepsilon^{-1} \exp(\tau^2/2\varepsilon) \int_{\lambda_i}^{\tau} \exp(-s^2/2\varepsilon) \cos[\omega(s - \lambda_i)]\, ds \tag{3.5}$$

As $\varepsilon \to 0$ and $\tau \gg \sqrt{2\varepsilon}$, we evaluate the integral asymptotically. The expression for (3.5) then becomes

$$y_1(\tau) \approx [\exp(\tau^2/2\varepsilon)] \sqrt{2\pi/\varepsilon}\, \cos(\omega\lambda_i) \tag{3.6}$$

which increases exponentially as $\tau > 0$. Thus, in order to keep the expansion in δ uniformly valid for $\tau > 0$, we consider exponentially small values of δ given by (2.10). The expansion (2.5) can be rewritten as

$$\begin{aligned} y \sim{} & A \exp[(\tau^2 - \lambda_i^2)/2\varepsilon] \\ & + \{\exp[(\tau^2 - k^2)/2\varepsilon]\}\, \delta_0(\varepsilon) \sqrt{2\pi/\varepsilon}\, \cos(\sigma\lambda_i/\varepsilon) + O(\delta^2) \end{aligned} \tag{3.7}$$

The expression (3.7) indicates that y increases exponentially as $\tau \to -\lambda_i$ ($\tau \to k$) if $k > -\lambda_i$ (if $k < -\lambda_i$). Furthermore, if $k < -\lambda_i$, we find from (3.7) that

$$y \sim \{\exp[(\tau^2 - k^2)/2\varepsilon]\} \sqrt{2\pi/\varepsilon}\, \cos(\sigma\lambda_i/\varepsilon) \qquad \text{as} \quad \tau \to k \tag{3.8}$$

As a consequence, $y \to \infty$ ($y \to -\infty$) if $\cos(\sigma\lambda_i/\varepsilon) > 0$ [$\cos(\sigma\lambda_i/\varepsilon) < 0$]. Furthermore, if $\cos(\sigma\lambda_i/\varepsilon)$ is exponentially small, the jump transition may appear at larger values than $\tau = k$.

3.3. $\sigma = O(\varepsilon^{1/2})$

This case appears because the asymptotic evaluation of the integral in (3.5) leads to a different result if $\omega = O(\varepsilon^{-1/2})$. We define a new frequency by

$$\sigma = \varepsilon^{1/2}\Omega \tag{3.9}$$

We now obtain from (3.5) the following expression for $y_1(\tau, \varepsilon)$:

$$y_1 \sim [\exp(\tau^2/2\varepsilon)]\sqrt{2\pi/\varepsilon}[\exp(-\Omega^2/2)]\cos(\Omega\lambda_i/\varepsilon^{1/2}) \tag{3.10}$$

which matches with (3.6) as $\Omega = \varepsilon^{1/2}\omega \to 0$. Consequently, the expansion (2.5) with (3.10) becomes

$$\begin{aligned} y \sim{}& A\exp[(\tau^2 - \lambda_i^2)/2\varepsilon] \\ &+ \{\exp[(\tau^2 - \sigma^2 - k^2)/2\varepsilon]\}\,\delta_0(\varepsilon)\sqrt{2\pi/\varepsilon}\cos(\sigma\lambda_i/\varepsilon) + O(\delta^2) \end{aligned} \tag{3.11}$$

By contrast to the previous case, the position of the jump transition is located at $\tau \sim (\sigma^2 + k^2)^{1/2}$ if $k < -\lambda_i$ and becomes a function of the frequency.

3.4. $\sigma = O(1)$

As σ approaches $O(1)$ quantities, (3.11) indicates that the first term may grow exponentially before the second term if $(\sigma^2 + k^2)^{1/2} > -\lambda_i$. Consequently, the jump transition will appear at $\tau = -\lambda_i$ and becomes independent of both δ and σ. This can be confirmed by a direct multitime analysis of Eqs. (1.2)–(1.4). Using the expansion (2.5), we then find that

$$y \sim A\exp[(\tau^2 - \lambda_i^2)/2\varepsilon] + \delta f(\tau, \sigma t) + O(\delta^2) \tag{3.12}$$

where f is a 2π-periodic function of σt. As τ increases, $y \to \infty$ as $\tau \to -\lambda_i$.

3.4. $\sigma \to \infty$

The limit $\sigma \to \infty$ follows the result for $\sigma = O(1)$, i.e., the expected transition occurs ar $\tau \sim -\lambda_i$. We study this case by a different multitime method. Defining $T = \sigma t$ as our fast time and t as our slow time, we seek as $\sigma \to \infty$ a solution of (1.2)–(1.4) in the form

$$y(T, t, \sigma) = y_0(T, t) + \sigma^{-1} y_1(T, t) + \cdots \tag{3.13}$$

After introducing (3.13) into (1.4) and equating to zero the coefficients of each power of σ^{-1}, we obtain a hierarchy of problems for y_0, y_1,.... By requiring bounded solutions with respect to T (solvability conditions), we obtain

$$y = y_0(t) + O(\sigma^{-1}) \tag{3.14}$$

where $y_0(t)$ satisfies the solvability condition:

$$y_{0t} = (\lambda_i + \varepsilon t)\, y_0 - y_0^3, \qquad y_0(0) = y_i \tag{3.15}$$

which is Eq. (2.6) if we define $\tau = \lambda_i + \varepsilon t$. The solution of this equation is given by (2.8) and reveals that the jump indeed appears at $\tau = -\lambda_i$. The $O(\sigma^{-1})$ correction in (3.14) involves periodic functions of T.

4. NUMERICAL STUDY

4.1. Constant Imperfection

In the first part of this section we analyze the effect of a constant imperfection. In Fig. 2, we represent the delay λ_j^2 as a function of the imper-

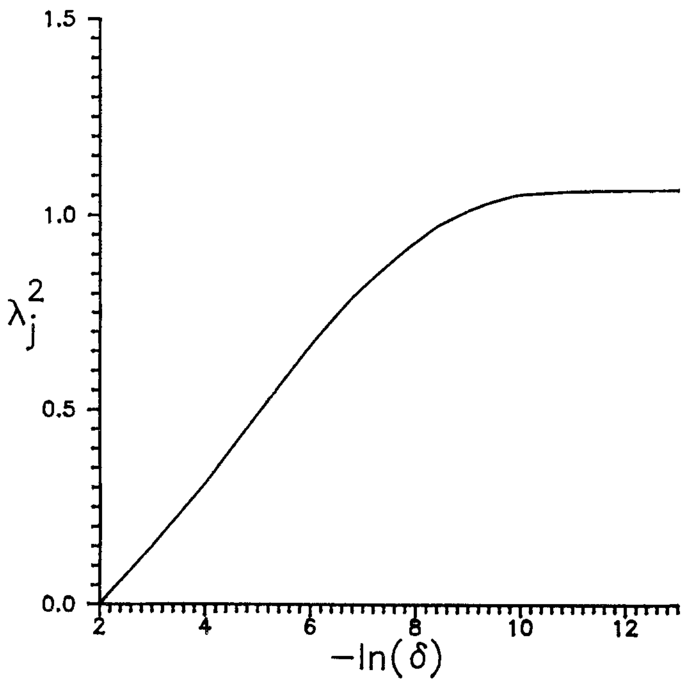

Fig. 2. Dependence of the delay on the imperfection. Parameters are $\varepsilon = 10^{-1}$, $y(0) = 0.5$, and $\lambda(0) = -1$.

fection parameter δ. The λ_j is numerically defined as the value at which $y(\lambda_j) = y_i$. The study of Eqs. (1.1)–(1.3) given in Section 2 suggests that there exist two distinct behaviors for $\lambda_j(\delta)$. First, as $\delta \to 0$, $\lambda_j^2 \sim \lambda_i^2$ and is independent of δ. The delay can only be controlled by the initial position of the bifurcation parameter. On the other hand, below a critical value $\delta_c(\varepsilon) \ll 1$, $\lambda_j^2 \sim k^2 = -2\varepsilon \ln \delta + O(\varepsilon \ln \varepsilon)$ and the delay now becomes a function of δ. Although our mathematical analysis is restricted to very small values of δ, these two regimes are clearly observed in Fig. 2. The first regime corresponds to the horizontal part of the line in Fig. 2; the second regime is associated with the inclined part of the line in Fig. 2 and its slope is approximatively equal to 2ε, as predicted by our analysis.

4.2. Time-Periodic Imperfections

We now analyze the effect of time-periodic imperfections. The results of our study are summarized in Table I. The values of the fixed parameters are $\lambda_i = -1$, $\varepsilon = 10^{-2}$, $\delta = 10^{-3}$. The value of the frequency σ is indicated in the table. It varies from small values ($\sigma = 10^{-3}$) to $O(1)$ values ($\sigma = 1$). If σ is sufficiently small, we expect to find the result of Section 3.1, i.e., the delay appears at $\lambda_j \sim k \sim (-2\varepsilon \ln \delta)^{1/2} \approx 0.37$. As σ progressively increases, λ_j increases continuously (Sections 3.2 and 3.3) and reaches its maximum when σ is sufficiently large (Section 3.4), i.e., $\lambda_j \sim -\lambda_i = 1$. An example of each case is displayed on Fig. 3.

Table I

σ	$\omega = \sigma/\varepsilon$	$\cos(\omega\lambda_i)$	Observed case	Prediction
10^{-3}	10^{-1}	+	$\sigma < \varepsilon$, Section 3.1	Minimum delay, $y \to \pm\infty$ if $y_i \gtrless 0$
10^{-2}	1	+	$\sigma = O(\varepsilon)$), Section 3.2	$y \to \pm\infty$ if $\cos(\omega\lambda_i) \gtrless 0$
5×10^{-2}	5	+		
10^{-1}	10	−		
1.5×10^{-1}	15	−	$\sigma = O(\varepsilon^{1/2})$, Section 3.3	Delay increases with σ: $\lambda_j \sim (\sigma^2 + k^2)^{1/2}$
2×10^{-1}	20	+		$\lambda_j \sim 0.42$
5×10^{-1}	50	+		$\lambda_j \sim 0.62$
5	500	+	$\sigma = O(1)$, Section 3.4	Maximum delay independent of σ and δ
10	1000	+		—

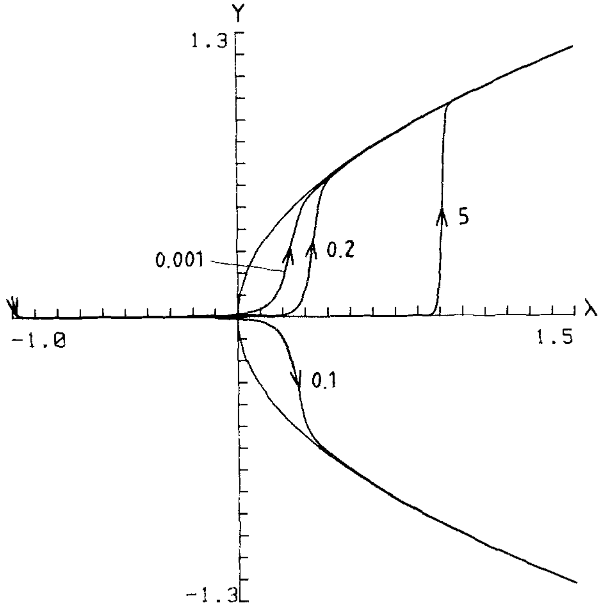

Fig. 3. Influence of the modulation frequency σ (given on the figure for each curve) on the solution of $y_t = \lambda(t)\,y - y^3 + \delta\cos(\sigma t)$. Parameters are $\delta = 10^{-3}$, $\lambda(t) = -1 + 10^{-2}t$. The curve $y^2 = \lambda$ is displayed for reference.

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